Lefschetz fixed point formula on a compact Riemannian manifold with boundary for some boundary conditions

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Outline

1. Introduction
2. de Rham complex $(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M), d)$ on a compact Riemannian manifold with boundary
3. Lefschetz fixed point formula on the complex $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$

Setting

- (M, Y, g^M) an m-dimensional compact oriented Riemannian manifold with boundary Y.
- g^M a product metric near the boundary Y.
- $f : M \to M$ a smooth map such that $f(Y) \subset Y$.
- A point $p \in M$ is a simple fixed point of f if $f(p) = p$ and $\det(I - df_p) \neq 0$.
  - The graph of f is transversal to the diagonal of $M \times M$ at (p, p).
  - Each simple fixed point is an isolated fixed point.
  - f has only finitely many fixed points.

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Boundary fixed points

- Let x be a boundary fixed point. Then $df_x : T_xM \to T_xM$ induces a map $df_{x,Y} : T_xY \to T_xY$.
- Denote $a_x = df_x (\text{mod}\, T_xY) : T_xM/T_xY \to T_xM/T_xY$.
- $a_x \geq 0$: T_xM/T_xY a normal half-line point inward at x.
- $a_x \neq 1$: since the fixed point x is simple.

Definition

A simple boundary fixed point x is called attracting if $a_x < 1$ and repelling if $a_x > 1$.

- $\mathcal{F}_0(f)$:= the set of all interior simple fixed points,
  $\mathcal{F}_Y^+(f)$:= the set of the attracting fixed points in Y
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Lefschetz fixed point formula

- Lefschetz fixed point formula for closed manifold M.

$$\sum_{q=0}^{m}(-1)^q \operatorname{Tr}(f^* : H^q(M) \to H^q(M)) = \sum_{p \in \mathcal{F}_0(f)} \operatorname{sgn}\det(I - df_p),$$

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Goal of this talk

- On the other hand, we introduced new de Rham complexes $\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0}(M), d\right)$ and $\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_1}(M), d\right)$ by using some boundary conditions $\widetilde{\mathcal{P}}_0$ and $\widetilde{\mathcal{P}}_1$.
- $H^q\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0}(M), d\right) = \begin{cases} H^q(M, Y) & \text{if} \quad q = \text{even} \\ H^q(M) & \text{if} \quad q = \text{odd} \end{cases}$
- $H^q\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_1}(M), d\right) = \begin{cases} H^q(M) & \text{if} \quad q = \text{even} \\ H^q(M, Y) & \text{if} \quad q = \text{odd} \end{cases}$
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Hodge decomposition on $\Omega^\bullet(Y)$

- $d^Y : \Omega^\bullet(Y) \to \Omega^\bullet(Y)$ the de Rham operator induced from $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$
- $\star_Y$ the Hodge star operator on Y induced from the Hodge star operator $\star_M$ on M
- $(d^Y)^*$ the formal adjoint of d^Y
- $\Delta_Y := (d^Y)^* d^Y + d^Y (d^Y)^*$ and $\mathcal{H}^\bullet(Y) := \operatorname{Ker} \Delta_Y$.
- The Hodge decomposition

$$\begin{aligned} \Omega^\bullet(Y) &= \operatorname{Im} d^Y \oplus \mathcal{H}^\bullet(Y) \oplus \operatorname{Im}(d^Y)^* \\ &= \left(\operatorname{Im} d^Y \oplus \mathcal{K}\right) \oplus \left(\star_Y \mathcal{K} \oplus \operatorname{Im}(d^Y)^*\right) \\ &= \Omega^\bullet_-(Y) \oplus \Omega^\bullet_+(Y), \end{aligned}$$

where $\mathcal{K}^q \cong \operatorname{Im}\left(\iota^* : H^q(M) \to H^q(Y)\right)$ such that $\iota : Y \hookrightarrow M$.

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Orthogonal projections $\mathcal{P}_{-,\mathcal{L}_0}, \mathcal{P}_{+,\mathcal{L}_1}$

- N : a collar neighborhood of Y which is isometric to $[0,1) \times Y$ and u : the coordinate normal to the boundary Y on N.
- On $\Omega^q(N)$, we identify $\omega_1 + du \wedge \omega_2$ with $\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$
- Define projections $\mathcal{P}_{-,\mathcal{L}_0}, \mathcal{P}_{+,\mathcal{L}_1} : \Omega^\bullet(Y) \oplus \Omega^\bullet(Y) \to \Omega^\bullet(Y) \oplus \Omega^\bullet(Y)$ by

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Boundary conditions for odd signature operator

- We define an involution $\Gamma : \Omega^q(M) \to \Omega^{m-q}(M)$ by

$$\Gamma\omega \quad := \quad i^{[\frac{m+1}{2}]}(-1)^{\frac{q(q+1)}{2}} \star_M \omega, \qquad \omega \in \Omega^q(M),$$

- The odd signature operator $\mathcal{B}$ acting on $\Omega^\bullet(M)$ is defined by

$$\mathcal{B} = d\Gamma + \Gamma d \quad : \quad \Omega^\bullet(M) \to \Omega^\bullet(M).$$

- $\mathcal{B}^2 = \Delta_M$, the Laplacian on M.
- $\mathcal{P}_{-,\mathcal{L}_0}$ and $\mathcal{P}_{+,\mathcal{L}_1}$ are well-posed boundary conditions for the odd signature opertator $\mathcal{B}$ in the sense of Seeley.
- If we restrict the domain of $\mathcal{B}$ to $\{\phi \in \Omega^\bullet(M) \mid \mathcal{P}_{-,\mathcal{L}_0}(\phi|_Y) = 0\}$ or $\{\phi \in \Omega^\bullet(M) \mid \mathcal{P}_{+,\mathcal{L}_1}(\phi|_Y) = 0\}$, $\mathcal{B}$ has compact resolvent and discrete spectra and is formally self-adjoint.

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de Rham complex $(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M), d)$

- To make de Rham complexes we define

$$\Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \quad = \quad \{\phi \in \Omega^q(M) \mid \mathcal{P}_{-,\mathcal{L}_0}\left(\left(\mathcal{B}^l \phi\right)|_Y\right) = 0,\ l = 0, 1, 2, \cdots\}.$$

- $\star_M \quad : \quad \Omega^{q,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \to \Omega^{m-q,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M)$
- The cochain complex $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M),\ d)$:

$$0 \longrightarrow \Omega^{0,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \xrightarrow{d} \Omega^{1,\infty}_{\mathcal{P}_{+,\mathcal{L}_1}}(M) \xrightarrow{d} \Omega^{2,\infty}_{\mathcal{P}_{-,\mathcal{L}_0}}(M) \xrightarrow{d} \cdots \longrightarrow 0.$$

- The cohomologies of the complex $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M),\ d)$ are

$$H^q\left(\Omega^{\bullet}_{\widetilde{\mathcal{P}}_0}(M), d\right) = \operatorname{Ker} \Delta^q_{\widetilde{\mathcal{P}}_0} = \begin{cases} H^q(M, Y) & \text{if} \quad q = \text{even} \\ H^q(M) & \text{if} \quad q = \text{odd} \end{cases}.$$

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Condition A

- For a smooth map $f : M \to M$, f is said to satisfy the Condition A if on some collar neighborhood $[0,\epsilon) \times Y$ of Y, $f : [0,\epsilon) \times Y \to M$ is expressed by

$$f(u, y) = (cu, B(y)),$$

where $c > 0$, $c \neq 1$ and $B : (Y, g^Y) \to (Y, g^Y)$ is an isometry.
- If $f : M \to M$ satisfies the Condition A, then
  - all the fixed points in Y are attracting if $0 < c < 1$ and repelling if $c > 1$.
  - for $\omega = \omega_1 + du \wedge \omega_2$ on a collar neighborhood of Y, $f^*\omega = B^*\omega_1 + cdu \wedge B^*\omega_2$.

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f^* preserves the complexes $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$

- Since B is an isometry, B^* maps $\operatorname{Im} d^Y$ onto $\operatorname{Im} d^Y$ and $\operatorname{Im}(d^Y)^*$ onto $\operatorname{Im}(d^Y)^*$.
- The following lemma shows that f^* preserves the complexes.

Lemma

B^* maps $\mathcal{K}^q$ onto $\mathcal{K}^q$ and $\star_Y\mathcal{K}^q$ onto $\star_Y\mathcal{K}^q$.

Proof.

The following commutative diagrams show that for $[\omega] \in H^q(M)$, $B^*\iota^*[\omega] = \iota^* f^*[\omega]$. Recall that $\iota^*[\omega] \in \mathcal{K}^q = \iota^*(H^q(M))$.

$$\begin{array}{ccc} Y & \xrightarrow{\iota} & M \\ \downarrow B & & \downarrow f \\ Y & \xrightarrow{\iota} & M \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} H^q(M) & \xrightarrow{\iota^*} & H^q(Y) \\ \downarrow f^* & & \downarrow B^* \\ H^q(M) & \xrightarrow{\iota^*} & H^q(Y) \end{array}$$

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$\square$

Lefschetz number of f on $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$

- Since f^* commutes with d, $f^* : (\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d) \to (\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\ d)$ is a cochain map.

Definition

Suppose that $f : M \to M$ is a smooth map satisfying the Condition A. We define the Lefschetz number of f with respect to the complex $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M),\ d)$ by

$$\begin{aligned} L_{\widetilde{\mathcal{P}}_0}(f) &= \sum_{q=0}^{m}(-1)^q \operatorname{Tr}\left(f^* : H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_i}(M),\ d)) \to H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_i}(M),\ d))\right) \\ &= \sum_{q=\text{even}} \operatorname{Tr}\left(f^* : H^q(M,Y) \to H^q(M,Y)\right) \\ &\quad - \sum_{q=\text{odd}} \operatorname{Tr}\left(f^* : H^q(M) \to H^q(M)\right) \end{aligned}$$

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Theorem

- (M, Y, g^M) : an m-dimensional compact oriented Riemannian manifold with boundary Y and g^M be a product metric near Y.
- $f : M \to M$ is a smooth map having only simple fixed points and satisfying the condition A.
- Then

$$L_{\widetilde{\mathcal{P}}_0}(f) = \sum_{x \in \mathcal{F}_0(f)} \operatorname{sgn}\det(I - df_x) + \frac{1}{2} \sum_{y \in \mathcal{F}_Y(f)} \operatorname{sgn}\det(I - df_y) - K_0$$

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where $K_0 = 0$ if B is orientation preserving and $K_0 = \operatorname{Tr}(B^* : \mathcal{K} \to \mathcal{K})$ if B is orientation reversing.

Lefschetz fixed point formula on $(\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0/\widetilde{\mathcal{P}}_1}(M),\, d)$

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Proof by heat kernel methods-1

$$
\begin{aligned}
L_{\widetilde{\mathcal{P}}_0}(f) &= \sum_{q=0}^{m} (-1)^q \operatorname{Tr}\left(f^* : H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M),\ d)) \to H^q((\Omega^{\bullet,\infty}_{\widetilde{\mathcal{P}}_0}(M), d)) \right) \\
&= \sum_{q=0}^{m} (-1)^q \operatorname{Tr}\left(f^* e^{-t\Delta^q_{\widetilde{\mathcal{P}}_0}} \right) \\
&= \lim_{t\to 0} \sum_{q=0}^{m} (-1)^q \operatorname{Tr}\left(f^* e^{-t\Delta^q_{\widetilde{\mathcal{P}}_0}} \right) \\
&= \lim_{t\to 0} \int_M \sum_{q=0}^{m} (-1)^q \operatorname{Tr}\left(\mathcal{T}_q(x) \mathcal{E}^q_{\widetilde{\mathcal{P}}_0}(t, f(x), x) \right) dvol(x),
\end{aligned}
$$

where $\mathcal{T}_q(x) := \Lambda^q df_x^T : \Lambda^q T^*_{f(x)} M \to \Lambda^q T^*_x M$ and $\mathcal{E}^q_{\widetilde{\mathcal{P}}_0}(t, x, z)$ is the kernel of $e^{-t\Delta^q_{\widetilde{\mathcal{P}}_0}}$.

Proof by heat kernel methods-2

For each $x \in \mathcal{F}_0(f)$, choose a small open neighborhood U_x of x such that $U_x \cap ([0,\epsilon) \times Y) = \emptyset$.

$$
\begin{aligned}
L_{\widetilde{\mathcal{P}}_0}(f) &= \lim_{t\to 0} \sum_{x\in\mathcal{F}_0(f)} \sum_{q=0}^{m} (-1)^q \int_{U_x} \mathrm{Tr}\left(\mathcal{T}_q(x)\mathcal{E}^q_{\widetilde{\mathcal{P}}_0}(t,f(x),x)\right) dvol(x) \\
&\quad + \lim_{t\to 0} \sum_{q=0}^{m} (-1)^q \int_Y \int_0^{\frac{\epsilon}{7}} \mathrm{Tr}\left(\mathcal{T}_q(x)\mathcal{E}^q_{\widetilde{\mathcal{P}}_0}(t,f(x),x)\right) du\, dvol(y) \\
&= (I) + (II)
\end{aligned}
$$

$$
(I) = \cdots = \sum_{x\in\mathcal{F}_0(f)} \operatorname{sgn}\det\left(I - df_x\right)
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Proof by heat kernel methods-3

$$
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(II) &= \lim_{t\to 0}\sum_{q=0}^{m}(-1)^q \int_Y \int_0^{\frac{\epsilon}{7}} \operatorname{Tr}\left(\mathcal{T}_q(x)\mathcal{E}^q_{\widetilde{\mathcal{P}}_0}(t,f(x),x)\right) du\, dvol(y) \\
&= \cdots \\
&= \frac{1-c}{2|1-c|}\cdot \lim_{t\to 0}\sum_{q=0}^{m-1}(-1)^q \operatorname{Tr}\left(B^* e^{-t\Delta_Y^q}\right) \\
&\quad + \lim_{t\to 0}\frac{1}{2}\sum_{q=0}^{m-1}\left(\operatorname{Tr}\left(B^* e^{-t\Delta_Y^q}|_{\Omega^q_+(Y)}\right) - \operatorname{Tr}\left(B^* e^{-t\Delta_Y^q}|_{\Omega^q_-(Y)}\right)\right)
\end{aligned}
$$

Proof by heat kernel methods-4

- $\operatorname{sgn}(1-c)\cdot\operatorname{sgn}\det(I-df_{x,Y}) = \operatorname{sgn}\det(I-df_x)$.
- The following diagram commutes

$$
\begin{tikzcd}
\operatorname{Im}(d^Y)^* \cap \Omega^q(Y) \arrow[r, "d^Y"] \arrow[d, "B^* e^{-t\Delta_Y}"] & \operatorname{Im} d^Y \cap \Omega^{q+1}(Y) \arrow[d, "B^* e^{-t\Delta_Y}"] \\
\operatorname{Im}(d^Y)^* \cap \Omega^q(Y) \arrow[r, "d^Y"] & \operatorname{Im} d^Y \cap \Omega^{q+1}(Y)
\end{tikzcd}
$$

-

$$
\begin{aligned}
(II) \quad = \quad & \frac{1}{2}\operatorname{sgn}\det(I-df_y) \\
& +\frac{1}{2}\left\{\operatorname{Tr}\left(B^* : (\star_Y\mathcal{K}) \to (\star_Y\mathcal{K})\right) - \operatorname{Tr}\left(B^* : \mathcal{K} \to \mathcal{K}\right)\right\}
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\mathrm{Im}(d^Y)^* \cap \Omega^q(Y) \arrow[r, "d^Y"] & \mathrm{Im}\, d^Y \cap \Omega^{q+1}(Y)
\end{tikzcd}$$

-

$$\begin{aligned}
(II) \quad = \quad & \frac{1}{2}\,\mathrm{sgn}\det(I - df_y) \\
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\end{aligned}$$

Thank you very much !